The Pre-Pub Paper

Glen Ritchie

January 8, 2026

Contents

List of Figures

1 Presentation

1.1 Preface

My name is Glen Ritchie. I have been a lifelong learner and student of Mathematics as a hobby.

I graduated from NDSU (2002) with a Bachelor of Science in Computer Science.

I am writing this to explain the many number triangles that I have found. Plus, there is a Number Triangle application written in Lazarus, a Pascal-derived language.

So why study number triangles? I've been playing with Number Theory for decades. It's been one of my primary interests. For instance, I wrote a paper on Pythagorean triples, which is available online.

Yet, my work on number triangles is novel.

Many people have enabled me to undertake this work, and I give glory to God for it. I thank my Lord Jesus.

Glen Ritchie

Bismarck, North Dakota - 2026

1.2 Introduction

The subject of my presentation is number triangles.

Why study number triangles? The Binomial numbers and their patterns in Pascal's Triangle[1] are an answer. (Or Binomial coefficients.)[2]

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}\,, n \geqslant k \geqslant 0 \tag{1}$$

Pascal's Triangle's addition rule is symbolized by Binomial numbers with this recurrence relation equation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{2}$$

How do you compute the Binomial numbers? Answer: It involves Factorials.

$$n! \tag{3}$$

I will not be studying Pascal's Triangle in depth, as Don Knuth notes in the next section. There are a couple of thousand known patterns of Binomial numbers and Pascal's Triangle.

I will be studying the field of number triangles.

I may need two meetings, to share most everything with you?

I have tools ranging from an explanation of how created, then a diagram, then a Windows application, plus an online dataset available to you.

1.3 Don Knuth Quote

A quote on Pascal's Triangle and Binomial numbers — discoveries.

This is from page 54 of the 3rd Edition (most recent, as of 2025) The Art of Computer Programming (TAOCP), Vol. 1, Fundamental Algorithms.[4]

> *Binomial coefficients satisfy literally thousands of identities,*[1] *and for centuries their amazing properties have been continually explored.*
>
> *In fact, there are so many relations present that when someone finds a new identity, not many people get excited about it any more, except for the discoverer.*

I hope to persuade you that there is still interesting material in the subject. There are many number triangles, with different patterns and types.

Even though Don Knuth's quote is discouraging. I understand; mathematics is vast. It is unlikely to find anything new. Yet, I have.

I aim to investigate patterns and structural symmetries in number triangles and determine whether these patterns can be connected to integer sequences in the OEIS database.

1.4 Pascal's Triangle

Pascal's Triangle has been realized, found, discovered, and rediscovered several times. It was known in China and the Islamic world before Pascal studied its properties and published it.

$$n = 5, k = 2, \binom{5}{2} = \frac{5!}{2! \cdot (5-2)!} = \frac{120}{2 \cdot 6} \tag{4}$$

Blaise Pascal collected results from several mathematicians who had discovered Pascal's Triangle. Then, he added more findings and realized their connection to Probability and Combinatorics. See Fig. 1, Pascal's Triangle on page 13. A Binonmial coefficient detailed illustration is Fig. 2, Pascal's Labeled Triangle, on page 14.

[1] An identity in mathematics is an equation that holds true for every permissible value of its variables.

1.4.1 RSD

- *Rising Shallow Diagonal Sum (RSD)*

What are they? Why is it important? How to calculate them?

I have a diagram, which shows how they are arranged. It starts on the left-hand side of a number triangle and slopes upwards to the right side. See Fig. 3, RSD Illustration Diagram on page 15.

Is there a Binomial formula? Yes. I don't intend to show it. Since, it isn't intuitive.

What are the applications of RSD values? Starting with it being an actual integer sequence.

1. They are building blocks for infinite series. You can derive an approximate series from an exact integer sequence. Then, apply it to a numerical calculation algorithm.
2. Data structures and runtime analyses (big-O notation) in Computer science, often produce connections to integer sequences.

1.4.2 RS

- *Row Sum (RS)*

The sum of a row, for each number triangle.

1.4.3 Number Triangle Format

- *Difference of Centered & Left-Justified Number Triangles*

The following diagram, shows the difference between a centered number triangle and a left-justified number triangle. See Fig. 4, Number Triangle Alignments on page 16.

I find the left-justified number triangle easier to read, in several ways. However, all my final number triangle diagrams are centered.

1.4.4 Simplex Numbers

- *Simplex Numbers & Simplices*

What are the Simplex numbers? Refers to a family of "Figurate numbers" when generalizing the Polygonal numbers.

1. Natural numbers
2. Triangular numbers

3. Tetrahedral numbers

4. Pentalope numbers

5. to ∞

Are there patterns to them, in their formulas? Yes. Successive expansions of:

$$n, n(n+1), n(n+1)(n+2), n(n+1)(n+2)(n+3), \ldots \tag{5}$$

$$\text{over} \quad 1!, 2!, 3!, 4!, \ldots \tag{6}$$

They are very special numbers in Pascal's Triangle. However, we don't see them often, beyond the Pentalope simplex numbers.

1.4.5 Geometric Relations

- zero-dimensional: Point.
- one-dimensional: Line segment.
- two-dimensional: Triangle.
- three-dimensional: Tetrahedron
- four-dimensional: called 4-simplex, or 5-cell Pentachoron.

1.5 NTA2 Application

There are several features to the Number Triangle Application (version) 2. A diagram of it's User Interface is shown in Fig. 5, NTA2 Application Interface From, on page 17.

- Allows you to create, store, retrieve, and make RSD / RS sums calculations on a relatively large number of triangles. • How many? Googol cubed times 1 trillion. Or $10^{100^3} \cdot 10^{12}$ If each cell holds a positive value from 1 to $10,000$?
- An open-source software application for Windows 10/11. Made with Lazarus, which is also an OSS development suite, based on the Pascal computer language.
- There are some form-filling wizards. The most powerful is the one that takes an integer sequence (18 terms) and produces a number triangle. Where the RSD sum equals the input values. The RS sum is where the surprise happens, when there is a connection.
- Extensive clipboard support is provided for the RSD and RS values. This is to assist you in searching OEIS's database for matches to the sums.

- Computes RSD Rising Shallow Diagonals, RS Row sums, and even FSD Falling Shallow Diagonals.
- There is even an option to shade the RSD fields gray when some number triangle elements are missing, indicating they are incorrect.
- Extensive cursor moving among the elements within the main form. Tabbing moves across, and the up/down cursor moves up/down.
- A visual cue is left when you change the value of an element. Although, if you enter an element, then leave it. There is also a color change, and this gives you a trial. Showing where you have been, in the number triangle elements.
- The file format saves more than the cells in the elements. It also saves the RSD & RS values, as well as the odd/even RSD and RS values. • How is this useful? It allows "text scanning" of `.tri` dataset files to search for numbers using any `grep`-like program, with your Operating System.

1.6 Twice-Filled Pascal's Triangle

- *Twice-Filled Pascal's Triangle*

What if? I would ask you to construct a triangular number array, with Fibonacci numbers as row sums and powers of 2 as the rising shallow diagonals?

Does it exist? What would it prove? What does it look like?

- See the Fig. 6, Pascal's Twice-Filled Number Triangle on page 18.
- A detailed illustration of how it is drawn, is Fig. 7, Pascal's Twice-Filled Revealed Number Triangle, on page 19.

Does this imply a different kind of power series produces values of Fibonacci numbers, not powers of 2? Or are the o.g.f.s merely transposed?

1.6.1 OEIS Presence

The twice-filled Pascal's Triangle pattern is listed in OEIS. It is sequence `https://oeis.org/A046854`. Described as follows:

- Triangle read by rows: $T(n, k) = binomial(floor((n + k)/2), k), n >= k >= 0$.

Is this a special number triangle? Perhaps, not. However, I find it very curious. It does exist on OEIS, and by the nature of symmetry with Pascal's Triangle. It flips the RSD and RS values.

1.7 Rudimentary Topics

1.7.1 OEIS

The On-line Encyclopedia of Integer Sequences at `https://www.oeis.org`.

It is a free database of integer sequences stored to reveal. What integer sequence connects to it?

How is it useful? It allows anyone to identify integer sequences without knowing their origins.

1.7.2 Git & GitHub

Git and GitHub are two separate products used for revision control and for organizing data repositories on GitHub.

Git is a tool for tracking changes in code or data and enabling collaboration.

There are several freely available Git GUI applications. SmartGit, in particular, allows unrestricted non-commercial use of its Git client software.

GitHub is a website where Git repositories are stored. The website fosters collaboration, with several features.

I encourage everyone to use these free tools and apply them on their computers. GitHub even allows "private" repositories to free users.

1.7.3 LaTeX

LaTeX is a typesetting system for document publication, with a focus on high-quality mathematical output.

There is so much that anyone can say here. However, I believe it is taught on this campus by the Department of Engineering.

If you are interested in an alternative? `Typst.app` appears to be very popular.

1.8 Teardrop Number Triangle

Why, the name Teardrop? Simple. The pattern of solving numbers tends to fall and then grow steadily.

1.8.1 Blackboard Challenge

What is the basis of Teardrop? A simple question.

- Is there a number triangle? Where the row sums (RS) don't matter and the rising shallow diagonal (RSD) equals the Triangular numbers? Can you find it?

1.8.2 Teardrop

While the objective for Teardrop is simple. Instead of the Fibonacci numbers being the solution to a Rising Shallow Diagonal? Is the goal to find the same result for Triangular numbers?

Is there such a triangle? Is it simple or complicated? Had no idea? Perhaps I was ignorant?

However, it isn't hard. And it exists. It's not listed on OEIS. From it, I discovered a new way to construct several number triangles.

Pause.

1.8.3 "On Board" Question

Explain the "on board" idea for the Introduction, as a teacher would.

Show, demonstrate, and talk about:

- Natural numbers
- Triangular numbers
- Pascal's Triangle: centered form.
- Pascal's Triangle: left-justified form.
- Then. Explain the challenge of Teardrop. Give someone a few minutes to guess.
- We can give them a hint with: _ = 1
- _ = 3
- _ + _ = 6
- _ + _ = 10, and so on.

1.8.4 Reveal of Teardrop Triangle

Here, it is. Almost beautiful. See Fig. 8, Teardrop Number Triangle on page 20.

1.8.5 History

Is this number triangle known to OEIS? It is not.

The following resources are very pertinent to understanding number triangles. [3], [7], [5], [6]

1.8.6 RSD Details

A lined Teardrop number triangle is shown in detail, in Fig. 9, Teardrop RSD Illustration Diagram, on page 21.

Attempting to understand. How do these RSD sums align with production of Triangular numbers? It is a very simple pattern. It is shown in Fig. 10, Reveal Teardrop Number Triangle, on page 22.

1.9 Vocabulary

Document for standards of capitalization, in my paper, presentations, or other material number triangle related, and reference.

- $\frac{1}{e}$
- Adobe InDesign
- Algebra
- asymptotic convergence
- Binomial numbers (coefficients)
- Construction Wizard
- continued fractions
- convoluted sequences, convolution
- difference tables
- directory, not folder
- e
- $e.g.f.$
- Engel expansion
- even bisection
- even numbers
- Factorial(s)
- Falling Shallow Diagonal • Ignoring standards
- Fibonacci numbers
- Figurate numbers
- generating function(s)
- Geometry
- Git
- GitHub
- Golden triangle, Golden number triangle
- Group Theory
- Hexagonal numbers, Concentric Hexagonal numbers
- Hexagonal Spiral pattern
- i
- Isosceles triangle
- $\LaTeX$
- Latin squares
- Lucas numbers
- Markdown article • Ignoring standards
- Markdown editor
- Math Club
- MathType
- MSE: Math Stack Exchange
- NT Layers (of Methods)
- Number Theory
- number sequence
- number triangle: symmetric, irregular, non-symmetric
- number triangle format: centered, left-justified
- odd bisection
- odd numbers
- OEIS: On-line Encyclopedia of Integer Sequences
- $o.g.f.$
- Padovan numbers
- Pascal's Triangle

- Pell numbers
- Pentagonal numbers, Generalized Pentagonal numbers
- Pentalope numbers
- pi, π
- Polygonal numbers
- Pre-Pub Document
- QuickStart Guide
- QR-codes
- reciprocal
- Row Sums • Ignoring standards.
- RSD: Rising Shallow Diagonal • Ignoring standards.
- Set Theory
- simplex
- 0-Simplex: Points.
- 1-Simplex: Line segments.
- 2-Simplex: Triangles.
- 3-Simplex: Tetrahedra.
- simplices
- SmartGit
- Teardrop triangle
- TeXstudio for Windows
- Triangular array
- Triangular numbers, sequence
- Triangular Spiral pattern
- Typora
- Ulam's Spiral pattern
- Wikipedia
- Zipper & Seam method
- Zotero

1.9.1 Common Triangles

1. Bell triangle
2. Bernoulli triangle
3. Catalan's triangle
4. Clark's triangle
5. Dudley triangle
6. Euler's number triangle
7. (Triangle of) Fibonomial Coefficient
8. Floyd's triangle
9. Fortaliza-Tetrahedral triangle
10. Golden triangle
11. Hosoya's triangle, (Fibonacci)
12. Losanitsch's triangle
13. Lucas triangle
14. Motzkin triangle
15. Narayana number triangle
16. Pascal's Triangle
17. Seidel-(Entringer-Arnold) triangle
18. Shapiro triangle
19. Tribonacci triangle

1.9.2 Common Sequences

- Bell numbers
- Catalan numbers
- Eulerian numbers, inside? Euler's n.t.
- Fibonacci numbers
- Jacobsthal numbers
- Lucas numbers
- Motzkin numbers
- Natural numbers
- Padovan numbers
- Pell numbers
- Tribonacci numbers

1.9.3 Alias: NT

- Number Theory
- Number triangle
- New Testament
- Nice Try (online)
- Northern Territory Australia, Northwest Territories Canada
- Windows New Technology
- No Thanks (online)
- Next Time (online)
- nucleotide

1.10 Figures

The following pages contain the diagrams of this manuscript.

References

[1] *Handbook of mathematical functions: With formulas, graphs, and mathematical tables*. Dover books on mathematics. Dover Publ, New York, NY, 9. dover print.; [nachdr. der ausg. von 1972] edition, 2013.

[2] Walter Gellert. *The VNR concise encyclopedia of mathematics*. Van Nostrand Reinhold Co, New York, 1st american ed. edition, 1977, 1975.

[3] Ronald L. Graham, Donald Ervin Knuth, and Oren Patashnik. *Concrete mathematics: A foundation for computer science*. Addison-Wesley, Reading Mass., 2nd ed. edition, 1994.

[4] Donald Ervin Knuth. *The art of computer programming*. Addison-Wesley, Reading Mass., 3rd ed. edition, 1997.

[5] Thomas Koshy. *Triangular arrays with applications*. Oxford University Press, Oxford and New York, 2011.

[6] Paul Barry. *Riordan Arrays: A Primer*. Logic Press, Ireland, first edition, 2016.

[7] Wikipedia. Figurate number: Gnomon.

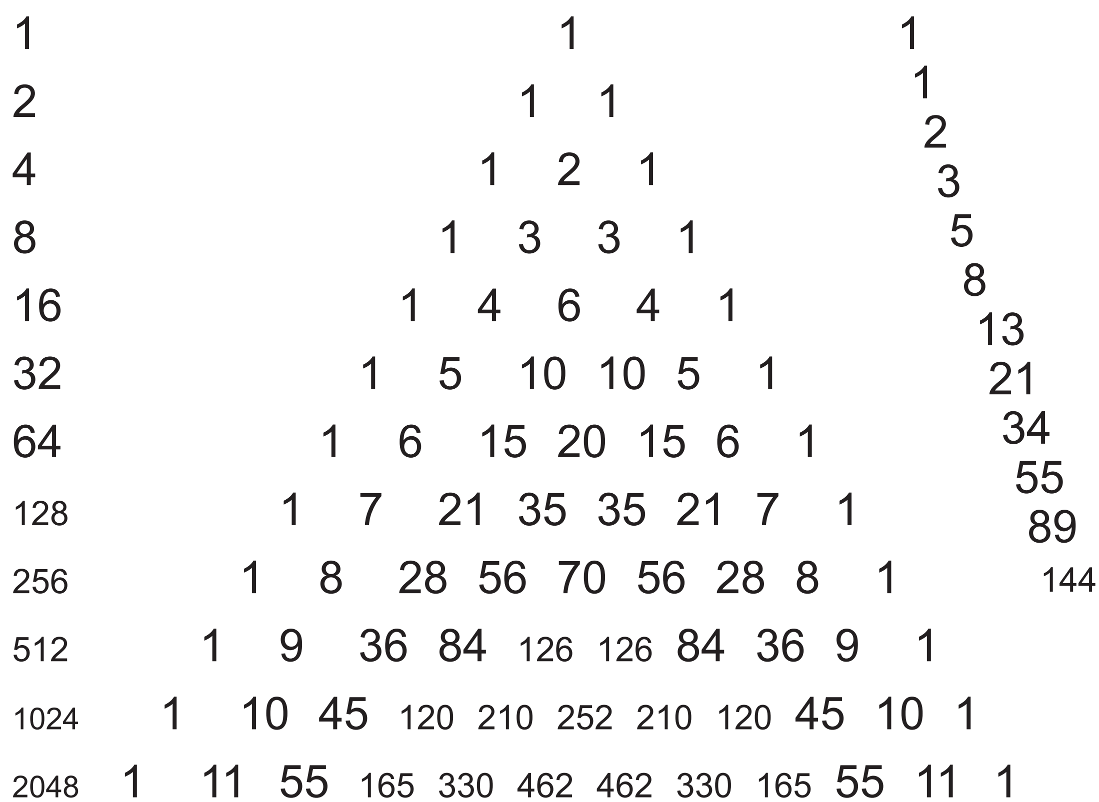

Figure 1: Pascal's Triangle

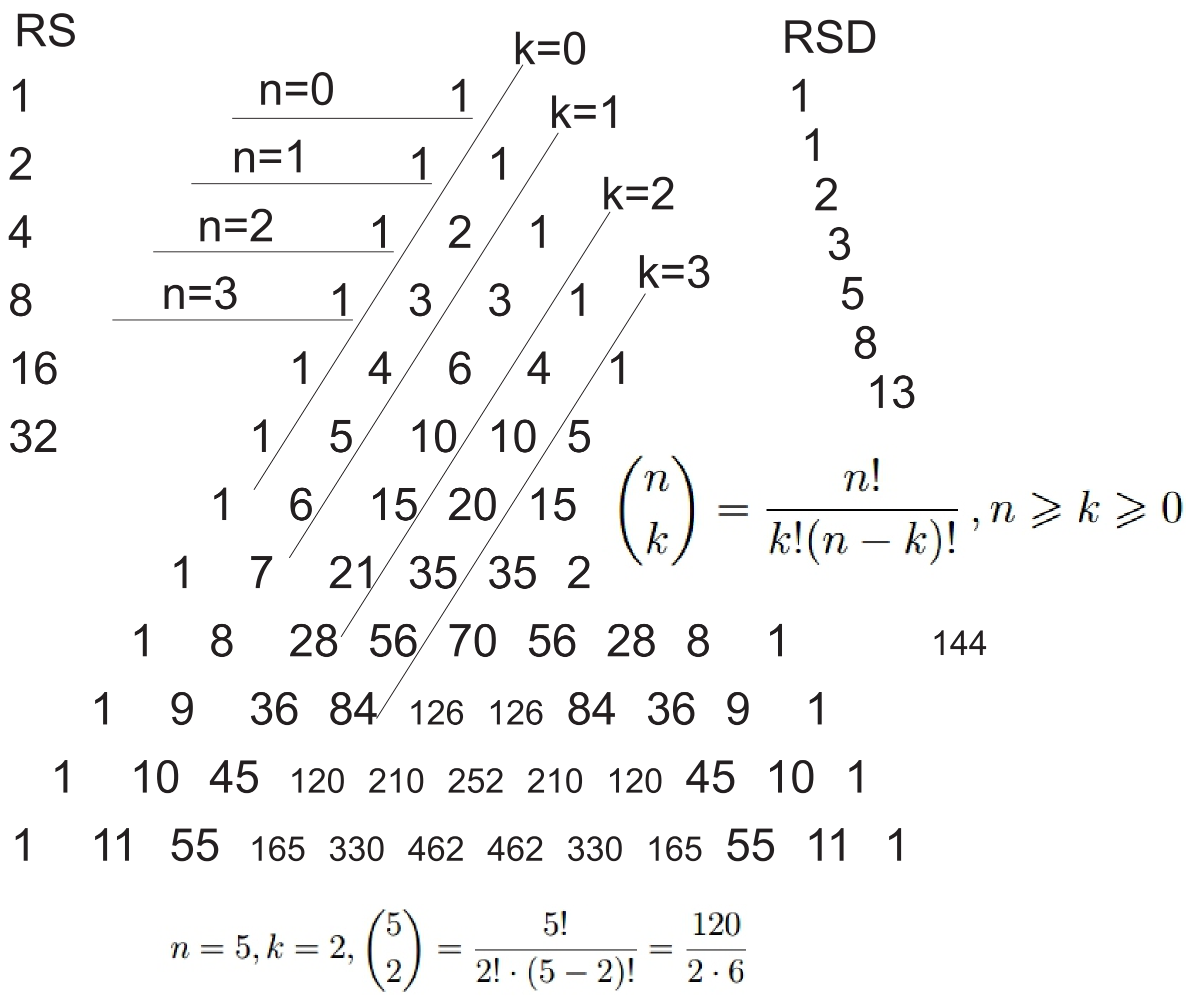

Figure 2: Pascal's Labeled Triangle

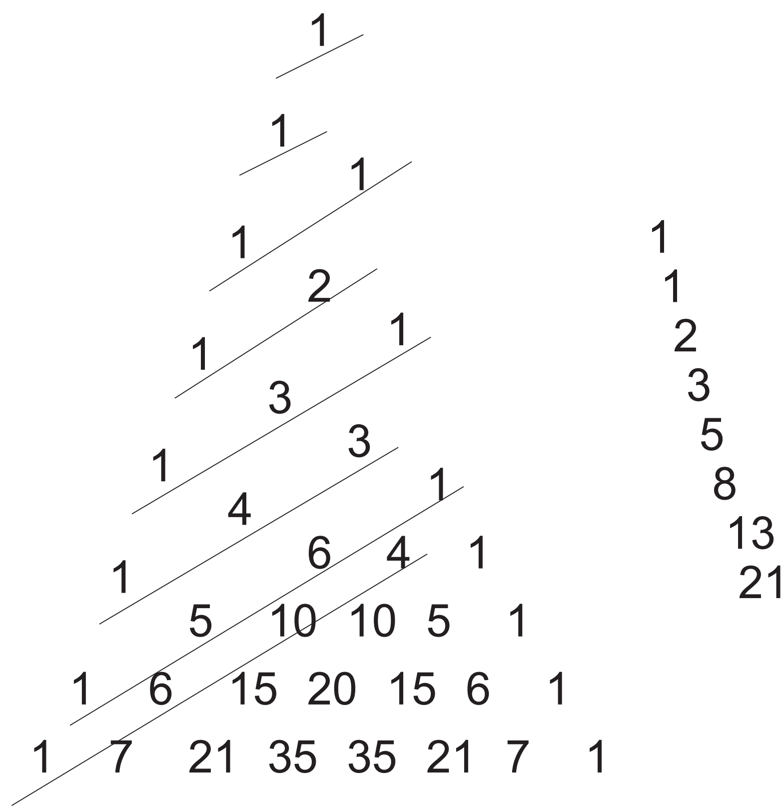

Figure 3: RSD Illustration Diagram

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Centered

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Left-Justified

Figure 4: Number Triangle Alignments

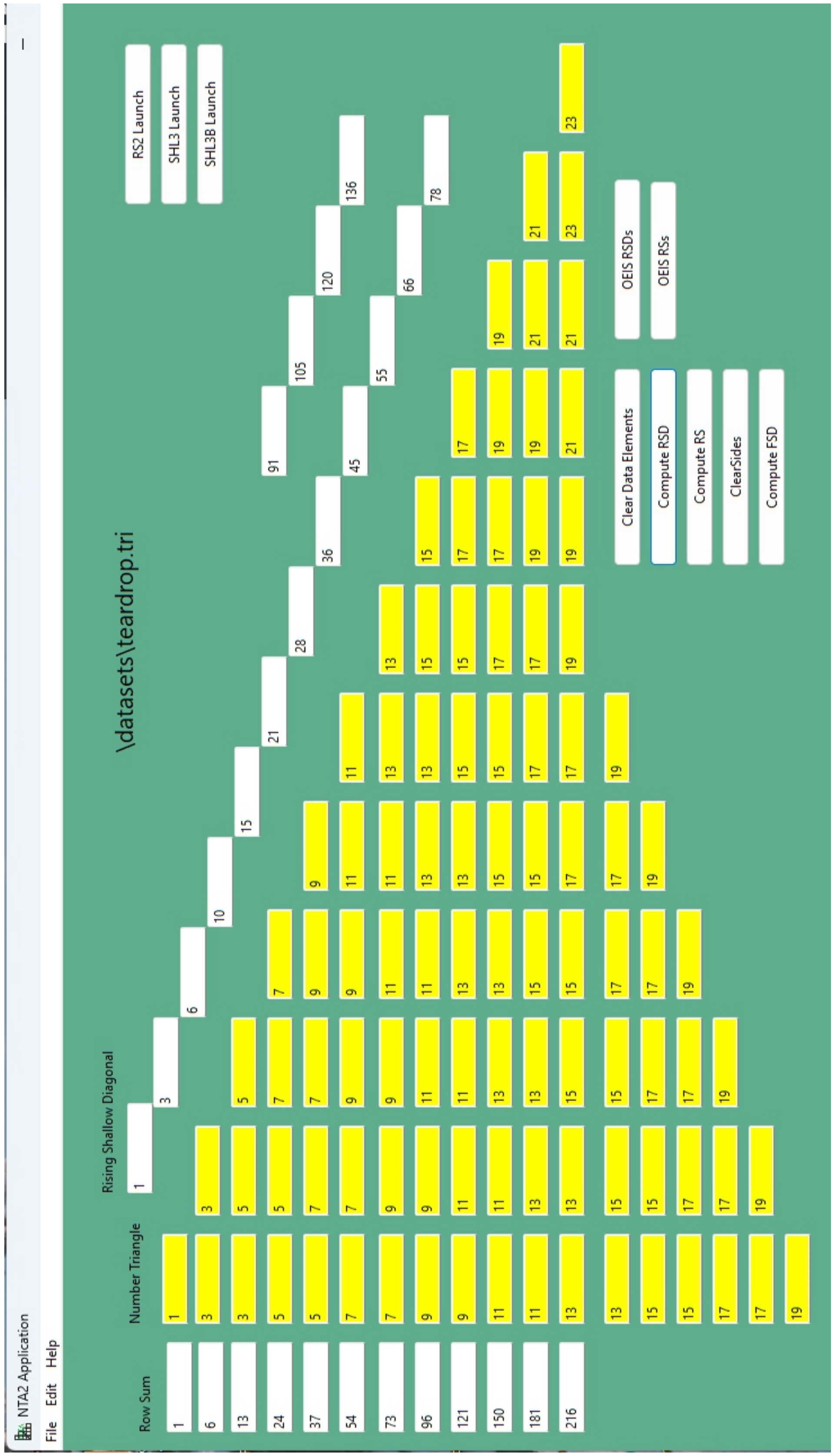

Figure 5: NTA2 Application Interface From

```
1                                   1                 1
2                                 1   1               1
3                               1   1   1             2
5                             1   2   1   1           2
8                           1   2   3   1   1         4
13                        1   3   3   4   1   1       4
21                      1   3   6   4   5   1   1     8
34                    1   4   6   10  5   6   1   1   8
55                  1   4   10  10  15  6   7   1   1   16
89                1   5   10  20  15  21  7   8   1   1   16
144             1   5   15  20  35  21  28  8   9   1   1   32
233           1   6   15  35  35  56  28  36  9   10  1   1   32
```

Figure 6: Pascal’s Twice-Filled Number Triangle

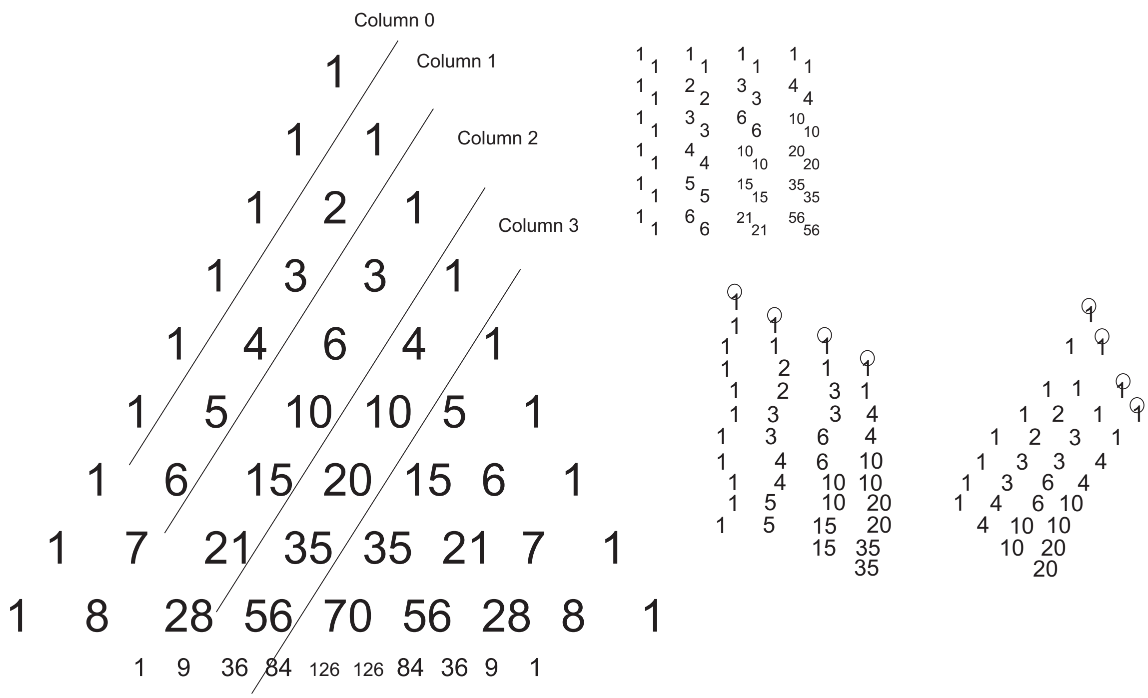

Figure 7: Pascal's Twice-Filled Revealed Number Triangle

1	1	1
6	3 3	3
13	3 5 5	6
24	5 5 7 7	10
37	5 7 7 9 9	15
54	7 7 9 9 11 11	21
73	7 9 9 11 11 13 13	28
96	9 9 11 11 13 13 15 15	36
121	9 11 11 13 13 15 15 17 17	45
150	11 11 13 13 15 15 17 17 19 19	55
181	11 13 13 15 15 17 17 19 19 21 21	66
216	13 13 15 15 17 17 19 19 21 21 23 23	78

Figure 8: Teardrop Number Triangle

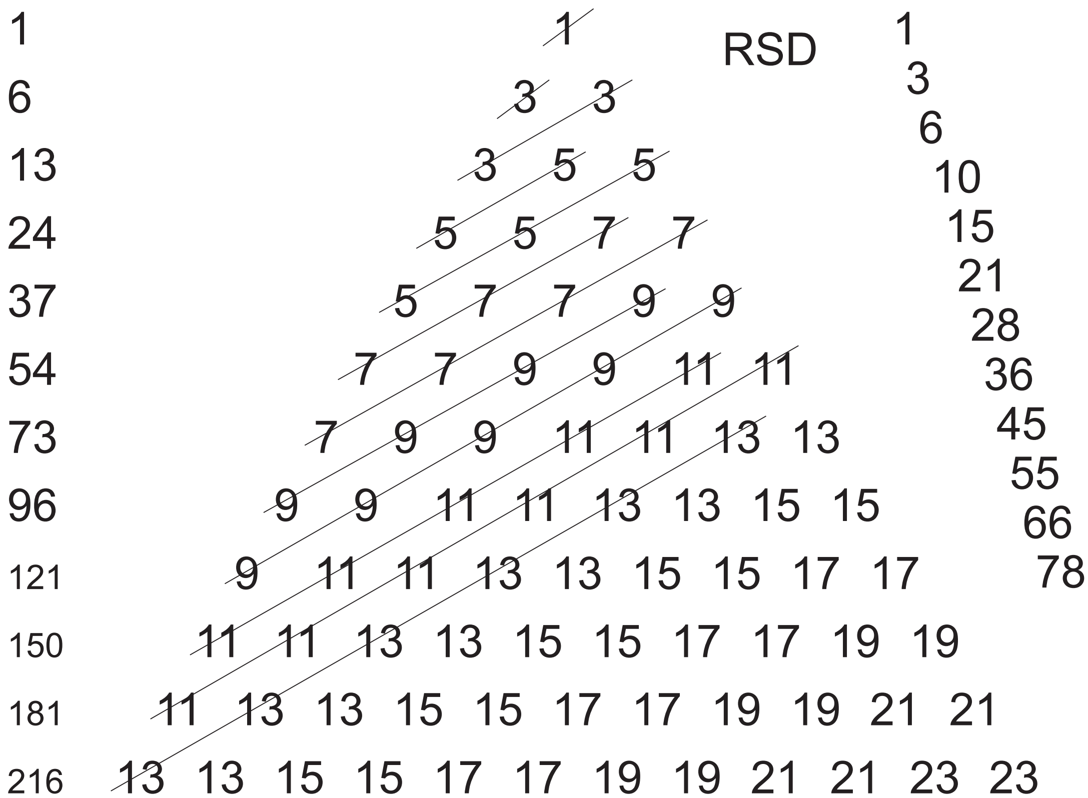

Figure 9: Teardrop RSD Illustration Diagram

```
1                        1              1
6                      3   3             3
13                   3   5   5            6
24                 5   5   7   7           10
37               5   7   7   9   9          15
54             7   7   9   9  11  11         21
73           7   9   9  11  11  13  13        28
96         9   9  11  11  13  13  15  15       36
121      9  11  11  13  13  15  15              45
                                                55
                                                66
                                                78
```

RSD Sums

1=1
3=3
3+3 = 6
5+5 = 10
5+5+5 = 15
7+7+7 = 21
7+7+7+7 = 28
9+9+9+9 = 36
9+9+9+9+9 = 45

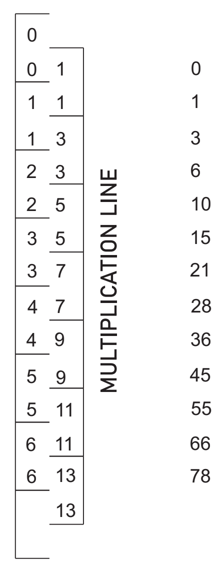

Figure 10: Reveal Teardrop Number Triangle